JEE(Advanced) – 2018 TEST PAPER - 2 WITH SOLUTION

(Exam Date: 20-05-2018)

PART-1 : MATHEMATICS

SECTION 1

1. For any positive integer n, define $f_n : (0, \infty) \to \mathbb{R}$ as

$$f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right) \text{ for all } x \in (0, \infty).$$

(Here, the inverse trigonometric function $\tan^{-1}x$ assume values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.)

Then, which of the following statement(s) is (are) TRUE ?

(A) $\sum_{j=1}^{5} \tan^2\left(f_j(0)\right) = 55$

(B) $\sum_{j=1}^{10} \left(1+f_j'(0)\right)\sec^2\left(f_j(0)\right) = 10$

(C) For any fixed positive integer n, $\lim_{x\to\infty} \tan\left(f_n(x)\right) = \frac{1}{n}$

(D) For any fixed positive integer n, $\lim_{x\to\infty} \sec^2\left(f_n(x)\right) = 1$

Ans. (D)

Sol. $f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{(x+j)-(x+j-1)}{1+(x+j)(x+j-1)}\right)$

$$f_n(x) = \sum_{j=1}^{n} [\tan^{-1}(x+j) - \tan^{-1}(x+j-1)]$$

$f_n(x) = \tan^{-1}(x + n) - \tan^{-1}x$

$\therefore \tan(f_n(x)) = \tan[\tan^{-1}(x + n) - \tan^{-1}x]$

$$\tan(f_n(x)) = \frac{(x+n)-x}{1+x(x+n)}$$

$$\tan(f_n(x)) = \frac{n}{1+x^2+nx}$$

$\therefore \ \sec^2(f_n(x)) = 1 + \tan^2(f_n(x))$

$$\sec^2(f_n(x)) = 1+\left(\frac{n}{1+x^2+nx}\right)^2$$

$$\lim_{x\to\infty}\sec^2(f_n(x)) = \lim_{x\to\infty}1+\left(\frac{n}{1+x^2+nx}\right)^2 = 1$$

2. Let T be the line passing through the points P(–2, 7) and Q(2, –5). Let F_1 be the set of all pairs of circles (S_1, S_2) such that T is tangents to S_1 at P and tangent to S_2 at Q, and also such that S_1 and S_2 touch each other at a point, say, M. Let E_1 be the set representing the locus of M as the pair (S_1, S_2) varies in F_1. Let the set of all straight line segments joining a pair of distinct points of E_1 and passing through the point R(1, 1) be F_2. Let E_2 be the set of the mid-points of the line segments in the set F_2. Then, which of the following statement(s) is (are) TRUE ?

(A) The point (–2, 7) lies in E_1

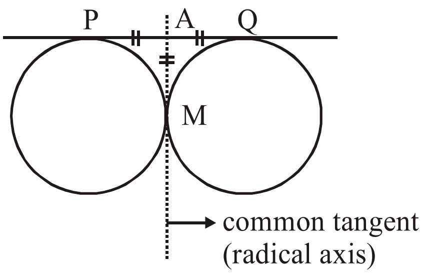
(B) The point $\left(\frac{4}{5},\frac{7}{5}\right)$ does **NOT** lie in E_2

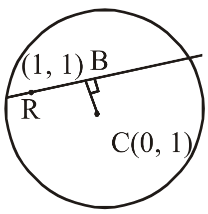
(C) The point $\left(\frac{1}{2},1\right)$ lies in E_2

(D) The point $\left(0,\frac{3}{2}\right)$ does **NOT** lie in E_1

Ans. (D)

Sol.





AP = AQ = AM

Locus of M is a circle having PQ as its diameter

Hence, $E_1 : (x - 2)(x + 2) + (y - 7)(y + 5) = 0$ and $x \neq \pm 2$

Locus of B (midpoint)

is a circle having RC as its diameter

$E_2 : x(x - 1) + (y - 1)^2 = 0$

Now, after checking the options, we get (D)

3. Let S be the of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_3 \in \mathbb{R}$ and the system of equations (in real variables)

$$-x + 2y + 5z = b_1$$
$$2x - 4y + 3z = b_2$$
$$x - 2y + 2z = b_3$$

has at least one solution. Then, which of the following system(s) (in real variables) has (have) at least one solution of each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$?

(A) $x + 2y + 3z = b_1$, $4y + 5z = b_2$ and $x + 2y + 6z = b_3$

(B) $x + y + 3z = b_1$, $5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$

(C) $-x + 2y - 5z = b_1$, $2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$

(D) $x + 2y + 5z = b_1$, $2x + 3z = b_2$ and $x + 4y - 5z = b_3$

Ans. (A,D)

Sol. We find D = 0 & since no pair of planes are parallel, so there are infinite number of solutions.

Let $\alpha P_1 + \lambda P_2 = P_3$

$\Rightarrow \quad P_1 + 7P_2 = 13P_3$

$\Rightarrow \quad b_1 + 7b_2 = 13b_3$

(A) $D \neq 0 \Rightarrow$ unique solution for any b_1, b_2, b_3

(B) $D = 0$ but $P_1 + 7P_2 \neq 13P_3$

(C) As planes are parallel and there exist infinite ordered triplet for which they will be non coincident although satisfying $b_1 + 7b_2 = 13b_3$.

$\therefore$ rejected.

(D) $D \neq 0$

4. Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = \frac{1}{2}$ and the parabola $y^2 = 4x$. Let these lines intersect at the point Q. Consider the ellipse whose center is at the origin O(0, 0) and whose semi-major axis is OQ. If the length of the minor axis of this ellipse is $\sqrt{2}$, then the which of the following statement(s) is (are) TRUE ?

(A) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is 1

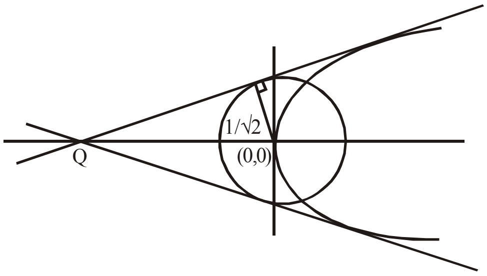
(B) For the ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$

(C) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and x = 1 is $\frac{1}{4\sqrt{2}}(\pi - 2)$

(D) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and x = 1 is $\frac{1}{16}(\pi - 2)$

Ans. **(A,C)**

Sol.



Let equation of common tangent is $y = mx + \frac{1}{m}$

$$\therefore \quad \left|\frac{0+0+\frac{1}{m}}{\sqrt{1+m^2}}\right| = \frac{1}{\sqrt{2}} \Rightarrow m^4 + m^2 - 2 = 0 \Rightarrow m = \pm 1$$

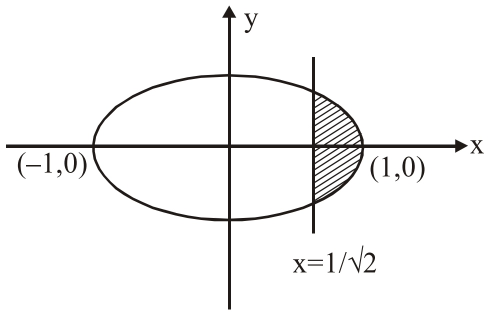
Equation of common tangents are $y = x + 1$ and $y = -x - 1$

point Q is $(-1, 0)$

$\therefore$ Equation of ellipse is $\frac{x^2}{1} + \frac{y^2}{1/2} = 1$

(A) $e = \sqrt{1-\frac{1}{2}} = \frac{1}{\sqrt{2}}$ and $LR = \frac{2b^2}{a} = 1$

(C)



$$\text{Area } \; 2.\int_{1/\sqrt{2}}^{1} \frac{1}{\sqrt{2}}.\sqrt{1-x^2}\,dx = \sqrt{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_{1/\sqrt{2}}^{1}$$

$$= \sqrt{2}\left[\frac{\pi}{4} - \left(\frac{1}{4} + \frac{\pi}{8}\right)\right] = \sqrt{2}\left(\frac{\pi}{8} - \frac{1}{4}\right) = \frac{\pi - 2}{4\sqrt{2}}$$

correct answer are (A) and (D)

5. Let s, t, r be the non-zero complex numbers and L be the set of solutions $z = x + iy$ $(x, y \in \mathbb{R}, i = \sqrt{-1})$ of the equation $sz + t\overline{z} + r = 0$, where $\overline{z} = x - iy$. Then, which of the following statement(s) is (are) TRUE ?

(A) If L has exactly one element, then $|s| \neq |t|$

(B) If $|s| = |t|$, then L has infinitely many elements

(C) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2

(D) If L has more than one element, then L has infinitely many elements

Ans. (A,C,D)

Sol. Given

$$sz + t\overline{z} + r = 0 \qquad (1)$$

$\overline{z} = x - iy$ (Conjugate of z)

Taking conjugate throughout $\overline{s}\overline{z} + \overline{t}z + \overline{r} = 0 \qquad (2)$

Adding (1) and (2)

$$\left(s + \overline{t}\right)z + \left(\overline{s} + t\right)\overline{z} + \left(r + \overline{r}\right) = 0$$

And Subtracting (1) and (2)

$$\left(s - \overline{t}\right)z + \left(t - \overline{s}\right)\overline{z} + \left(r - \overline{r}\right) = 0$$

For unique solution

$$\frac{t + \overline{s}}{t - s} \neq \frac{s + \overline{t}}{s - \overline{t}}$$

On further simplification $\Rightarrow |t| \neq |s|$

Hence option A proved.

If the lines coincide, then

$$\frac{t + \overline{s}}{t - \overline{s}} = \frac{\overline{t} + s}{s - \overline{t}} = \frac{r + \overline{r}}{r - \overline{r}}$$

On comparing

$$\frac{t + \overline{s}}{t - \overline{s}} = \frac{r + \overline{r}}{r - \overline{r}}$$

and simplification, we get $\Rightarrow |s| = |t|$

The lines can be parallel or coincidental.

Since, no concrete outcome.

Hence, option B is not correct.

Clearly L is either a single or represents a line and $|z-1+i|=5$ represents a circle.

$\therefore$ Intersection of L and $\{|z-1+i|=5\}$ is ATMOST 2.

Hence, option C is correct.

Let $s=\alpha_1+i\beta_1;\ t=\alpha_2+i\beta_2\ and\ r=\alpha_3+i\beta_3$

Then $sz+t\bar{z}+r=0$

$$\Rightarrow(\alpha_1+\alpha_2)x+(\beta_2-\beta_1)y+\alpha_3=0$$
$$and\quad(\beta_1+\beta_2)x+(\alpha_1-\alpha_2)y+\beta_3=0$$

If L has more than 1 element then it implies L will have $\propto$ elements.

As L represents linear equation in x and y.

Hence, option D is correct.

6. Let $f:(0,\pi)\to\mathbb{R}$ be a twice differentiable function such that

$$\lim_{t\to x}\frac{f(x)\sin t-f(t)\sin x}{t-x}=\sin^2 x \text{ for all } x\in(0,\pi).$$

If $f\left(\frac{\pi}{6}\right)=-\frac{\pi}{12}$, then which of the following statement(s) is (are) TRUE ?

(A) $f\left(\frac{\pi}{4}\right)=\frac{\pi}{4\sqrt{2}}$

(B) $f(x)<\frac{x^4}{6}-x^2$ for all $x\in(0,\pi)$

(C) There exists $\alpha\in(0,\pi)$ such that $f'(\alpha)=0$

(D) $f''\left(\frac{\pi}{2}\right)+f\left(\frac{\pi}{2}\right)=0$

Ans. (B,C,D)

Sol. $\lim_{t\to x}\frac{f(x)\sin t - f(t)\sin x}{t-x} = \sin^2 x$

by using L'Hopital

$\lim_{t\to x}\frac{f(x)\cos t - f'(t)\sin x}{1} = \sin^2 x$

$\Rightarrow$ f(x)cosx –f'(x)sinx = $\sin^2$x

$\Rightarrow \quad -\left(\frac{f'(x)\sin x - f(x)\cos x}{\sin^2 x}\right) = 1$

$\Rightarrow \quad -d\left(\frac{f(x)}{\sin x}\right) = 1$

$\Rightarrow \quad \frac{f(x)}{\sin x}$=-x+c

Put $x = \frac{\pi}{6}$ & $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$

$\therefore$ c = 0 $\Rightarrow$ f(x) = –xsinx

(A) $f\left(\frac{\pi}{4}\right) = \frac{-\pi}{4}\frac{1}{\sqrt{2}}$

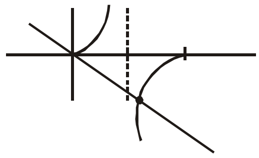
(B) f(x) = –x sinx

as sinx > $x - \frac{x^3}{6}$, –x sinx < – x^2 + $\frac{x^4}{6}$

$\therefore$ f(x) < – x^2 + $\frac{x^4}{6}$ $\forall$ x $\in$ (0, π)

(C) f'(x) = –sinx – x cosx

f'(x) = 0 $\Rightarrow$ tanx = –x $\Rightarrow$ there exist $\alpha \in (0, \pi)$ for which $f'(\alpha) = 0$



(D) f''(x) = –2cosx + xsinx

$f''\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$, $f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$

$f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$

SECTION 2

7. The value of the integral

$$\int_0^{\frac{1}{2}} \frac{1+\sqrt{3}}{\left((x+1)^2(1-x)^6\right)^{\frac{1}{4}}}dx$$

is ______ .

Ans. (2)

Sol. $$\int_0^{\frac{1}{2}} \frac{\left(1+\sqrt{3}\right)dx}{\left[(1+x)^2(1-x)^6\right]^{1/4}}$$

$$\int_0^{\frac{1}{2}} \frac{\left(1+\sqrt{3}\right)dx}{(1+x)^2\left[\frac{(1-x)^6}{(1+x)^6}\right]^{1/4}}$$

Put $\frac{1-x}{1+x} = t \Rightarrow \frac{-2dx}{(1+x)^2} = dt$

$$I = \int_1^{1/3} \frac{\left(1+\sqrt{3}\right)dt}{-2t^{6/4}} = \frac{-(1+\sqrt{3})}{2} \times \left|\frac{-2}{\sqrt{t}}\right|_1^{1/3} = \left(1+\sqrt{3}\right)\left(\sqrt{3}-1\right) = 2$$

8. Let P be a matrix of order 3×3 such that all the entries in P are from the set $\{-1, 0, 1\}$. Then, the maximum possible value of the determinant of P is ______ .

Ans. (4)

Sol. $$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \underbrace{(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2)}_{x} - \underbrace{(a_3b_2c_1 + a_2b_1c_3 + a_1b_3c_2)}_{y}$$

Now if $x \le 3$ and $y \ge -3$

the Δ can be maximum 6

But it is not possible

as $x = 3 \Rightarrow$ each term of $x = 1$

and $y = 3 \Rightarrow$ each term of $y = -1$

$$\Rightarrow \prod_{i=1}^{3} a_ib_ic_i = 1 \text{ and } \prod_{i=1}^{3} a_ib_ic_i = -1$$

which is contradiction

so now next possibility is 4

which is obtained as $\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 1(1+1)-1(-1-1)+1(1-1) = 4$

9. Let X be a set with exactly 5 elements and Y be a set with exactly 7 elements. If α is the number of one-one functions from X to Y and β is the number of onto functions from Y to X, then the value of $\frac{1}{5!}(\beta-\alpha)$ is _____ .

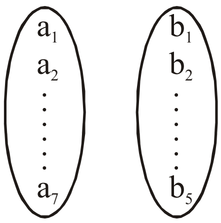
Ans. (119)

Sol. $n(X) = 5$

$n(Y) = 7$

$\alpha \to$ Number of one-one function = ${}^7C_5 \times 5!$

$\beta \to$ Number of onto function Y to X



1, 1, 1, 1, 3   1, 1, 1, 2, 2

$\frac{7!}{3!4!} \times 5! + \frac{7!}{(2!)^3 3!} \times 5! = \left({}^7C_3 + 3.{}^7C_3\right)5! = 4 \times {}^7C_3 \times 5!$

$\frac{\beta-\alpha}{5!} = 4 \times {}^7C_3 - {}^7C_5 = 4 \times 35 - 21 = 119$

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 0$. If $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = (2+5y)(5y-2),$$

then the value of $\lim_{x \to -\infty} f(x)$ is _____ .

Ans. (0.4)

Sol. $\frac{dy}{dx} = 25y^2 - 4$

So, $\frac{dy}{25y^2-4} = dx$

Integrating, $\frac{1}{25} \times \frac{1}{2 \times \frac{2}{5}} \ln\left|\frac{y-\frac{2}{5}}{y+\frac{2}{5}}\right| = x + c$

$$\Rightarrow \quad \ln\left|\frac{5y-2}{5y+2}\right| = 20(x+c)$$

Now, c = 0 as f(0) = 0

Hence $\left|\frac{5y-2}{5y+2}\right| = e^{(20x)}$

$$\operatorname*{let}_{x\to-\infty}\left|\frac{5f(x)-2}{5f(x)+2}\right| = \operatorname*{let}_{x\to-\infty} e^{(20x)}$$

Now, RHS = 0 $\Rightarrow \operatorname*{let}_{x\to-\infty}(5f(x)-2) = 0$

$$\Rightarrow \operatorname*{let}_{x\to-\infty} f(x) = \frac{2}{5}$$

11. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 1$ and satisfying the equation

$$f(x + y) = f(x)f'(y) + f'(x)f(y) \text{ for all } x, y \in \mathbb{R}.$$

Then, then value of $\log_e(f(4))$ is _____ .

Ans. **(2)**

Sol. P(x, y) : $f(x + y) = f(x)f'(y) + f'(x) f(y) \ \forall\ x, y \in R$

P(0, 0) : $f(0) = f(0)f'(0) + f'(0) f(0)$

$\Rightarrow 1 = 2f'(0)$

$$\Rightarrow f'(0) = \frac{1}{2}$$

P(x, 0) : $f(x) = f(x).\ f'(0) + f'(x).f(0)$

$$\Rightarrow f(x) = \frac{1}{2}f(x) + f'(x)$$

$$\Rightarrow f'(x) = \frac{1}{2}f(x)$$

$$\Rightarrow f(x) = e^{\frac{1}{2}x}$$

$\Rightarrow \ln(f(4)) = 2$

12. Let P be a point in the first octant, whose image Q in the plane x + y = 3 (that is, the line segment PQ is perpendicular to the plane x + y = 3 and the mid-point of PQ lies in the plane x + y = 3) lies on the z-axis. Let the distance of P from the x-axis be 5. If R is the image of P in the xy-plane, then the length of PR is _____ .

Ans. **(8)**

Sol. Let $P(\alpha, \beta, \gamma)$

$Q(0, 0, \gamma)$ &

$R(\alpha, \beta, -\gamma)$

Now, $\overline{PQ} \parallel \hat{i} + \hat{j} \Rightarrow \left(\alpha\hat{i} + \beta\hat{j}\right) \parallel \left(\hat{i} + \hat{j}\right)$

$\Rightarrow \quad \alpha = \beta$

Also, mid point of PQ lies on the plane $\Rightarrow \dfrac{\alpha}{2} + \dfrac{\beta}{2} = 3 \Rightarrow \alpha + \beta = 6 \Rightarrow \alpha = 3$

Now, distance of point P from X-axis is $\sqrt{\beta^2 + \gamma^2} = 5$

$\Rightarrow \quad \beta^2 + \gamma^2 = 25 \Rightarrow \gamma^2 = 16$

as $\beta = \alpha = 3$

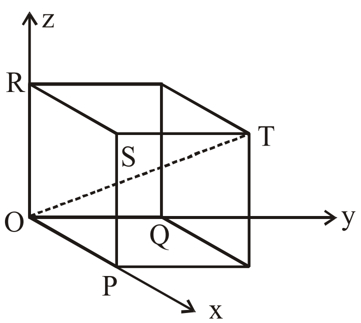
as $\gamma = 4$

Hence, $PR = 2\gamma = 8$

13. Consider the cube in the first octant with sides OP, OQ and OR of length 1, along the x-axis, y-axis and z-axis, respectively, where O(0, 0, 0) is the origin. Let $S\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ be the centre of the cube and T be the vertex of the cube opposite to the origin O such that S lies on the diagonal OT. If $\vec{p} = \overrightarrow{SP}$, $\vec{q} = \overrightarrow{SQ}$, $\vec{r} = \overrightarrow{SR}$ and $\vec{t} = \overrightarrow{ST}$, then the value of $\left|(\vec{p} \times \vec{q}) \times (\vec{r} \times \vec{t})\right|$ is _____ .

Ans. **(0.5)**

Sol.



$\vec{p} = \overrightarrow{SP} = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{2}\left(\hat{i} - \hat{j} - \hat{k}\right)$

$$\vec{q}=\overrightarrow{SQ}=\left(-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)=\frac{1}{2}\left(-\hat{i}+\hat{j}-\hat{k}\right)$$

$$\vec{r}=\overrightarrow{SR}=\left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)=\frac{1}{2}\left(-\hat{i}-\hat{j}+\hat{k}\right)$$

$$\vec{t}=\overrightarrow{ST}=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)=\frac{1}{2}\left(\hat{i}+\hat{j}+\hat{k}\right)$$

$$\left|(\vec{p}\times\vec{q})\times(\vec{r}\times\vec{t})\right|=\frac{1}{4}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & -1\\ -1 & 1 & -1\end{vmatrix}\times\frac{1}{4}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & -1 & 1\\ 1 & 1 & 1\end{vmatrix}$$

$$=\frac{1}{16}\left|\left(2\hat{i}+2\hat{j}\right)\times\left(-2\hat{i}+2\hat{j}\right)\right|=\left|\frac{\hat{k}}{2}\right|=\frac{1}{2}$$

14. Let $X=\left({}^{10}C_1\right)^2+2\left({}^{10}C_2\right)^2+3\left({}^{10}C_3\right)^2+...+10\left({}^{10}C_{10}\right)^2$, where ${}^{10}C_r$, $r \in \{1, 2, ..., 10\}$ denote binomial coefficients. Then, the value of $\frac{1}{1430}X$ is _____ .

Ans. (646)

Sol. $X=\sum_{r=0}^{n} r.\left({}^{n}C_r\right)^2; n=10$

$$X=n.\sum_{r=0}^{n} {}^{n}C_r.{}^{n-1}C_{r-1}$$

$$X=n.\sum_{r=1}^{n} {}^{n}C_{n-r}.{}^{n-1}C_{r-1}$$

$$X=n.{}^{2n-1}C_{n-1};\ n=10$$

$$X=10.{}^{19}C_9$$

$$\frac{X}{1430}=\frac{1}{143}.{}^{19}C_9$$

$= 646$

SECTION 3

15. Let $E_1 = \left\{x \in \mathbb{R} : x \neq 1 \text{ and } \frac{x}{x-1} > 0\right\}$

and $E_2 = \left\{x \in E_1 : \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right) \text{ is a real number}\right\}$.

$\left(\text{Here, the inverse trigonometric function } \sin^{-1}x \text{ assumes values in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].\right)$

Let $f : E_1 \to \mathbb{R}$ be the function defined by $f(x) = \log_e\left(\frac{x}{x-1}\right)$

and $g : E_2 \to \mathbb{R}$ be the function defined by $g(x) = \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right)$.

LIST-I	**LIST-II**
P. The range of f is	**1.** $\left(-\infty, \frac{1}{1-e}\right] \cup \left[\frac{e}{e-1}, \infty\right)$
Q. The range of g contains	**2.** $(0, 1)$
R. The domain of f contains	**3.** $\left[-\frac{1}{2}, \frac{1}{2}\right]$
S. The domain of g is	**4.** $(-\infty, 0) \cup (0, \infty)$
	5. $\left(-\infty, \frac{e}{e-1}\right]$
	6. $(-\infty, 0) \cup \left(\frac{1}{2}, \frac{e}{e-1}\right]$

The correct option is :

(A) **P → 4; Q → 2; R → 1; S → 1**

(B) **P → 3; Q → 3; R → 6; S → 5**

(C) **P → 4; Q → 2; R → 1; S → 6**

(D) **P → 4; Q → 3; R → 6; S → 5**

Ans. (A)

Sol. $E_1 : \dfrac{x}{x-1} > 0$

$$\begin{array}{ccccc} + & & - & & + \\ \hline & 0 & & 1 & \end{array}$$

$\Rightarrow \quad E_1 : x \in (-\infty, 0)$ □ $\cup (1, \infty)$

$E_2 : -1 \le \ell n\left(\dfrac{x}{x+1}\right) \le 1$

$$\frac{1}{e} \le \frac{x}{x-1} \le e$$

Now $\dfrac{x}{x-1} - \dfrac{1}{e} \ge 0$

$\Rightarrow \quad \dfrac{(e-1)x+1}{e(x-1)} \ge 0$

$$\begin{array}{ccccc} + & & - & & + \\ \hline & -1/(e-1) & & 1 & \end{array}$$

$\Rightarrow \quad x \in \left(-\infty, \dfrac{1}{1-e}\right] \cup (1, \infty)$

also $\quad \dfrac{x}{x-1} - e \le 0$

$\dfrac{(e-1)x-e}{x-1} \ge 0$

$$\begin{array}{ccccc} + & & - & & + \\ \hline & 1 & & e/(e-1) & \end{array}$$

$\Rightarrow \quad x \in (-\infty, 1) \cup \left[\dfrac{e}{e-1}, \infty\right]$

So $\quad E_2 : \left(-\infty, \dfrac{1}{1-e}\right) \cup \left[\dfrac{e}{e-1}, \infty\right]$

as Range of $\dfrac{x}{x-1}$ is $R^+ - \{1\}$

$\Rightarrow \quad$ Range of f is $R - \{0\}$ or $(-\infty, 0) \cup (0, \infty)$

Range of g is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$ or $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

Now P $\rightarrow$ 4, Q $\rightarrow$ 2, R $\rightarrow$ 1, S $\rightarrow$ 1

Hence A is correct

16. In a high school, a committee has to be formed from a group of 6 boys M_1, M_2, M_3, M_4, M_5, M_6 and 5 girls G_1, G_2, G_3, G_4, G_5.

(i) Let α_1 be the total number of ways in which the committee can be formed such that the committee has 5 members, having exactly 3 body and 2 girls.

(ii) Let α_2 be the total number of ways in which the committe can be formed such that the committee has at least 2 members, and having an equal number of boys and girls.

(iii) Let α_3 be the total number of ways in which the committe can be formed such that the committee has 5 members, at least 2 of them being girls.

(iv) Let α_4 be the total number of ways in which the committee can be formed such that the commitee has 4 members, having at least 2 girls and such that both M_1 and G_1 are **NOT** in the committee together.

LIST-I	**LIST-II**
P. The value of α_1 is	**1.** 136
Q. The value of α_2 is	**2.** 189
R. The value of α_3 is	**3.** 192
S. The value of α_4 is	**4.** 200
	5. 381
	6. 461

The correct option is :-

(A) **P $\rightarrow$ 4; Q $\rightarrow$ 6, R $\rightarrow$ 2; S $\rightarrow$ 1**

(B) **P $\rightarrow$ 1; Q $\rightarrow$ 4; R $\rightarrow$ 2; S $\rightarrow$ 3**

(C) **P $\rightarrow$ 4; Q $\rightarrow$ 6, R $\rightarrow$ 5; S $\rightarrow$ 2**

(D) **P $\rightarrow$ 4; Q $\rightarrow$ 2; R $\rightarrow$ 3; S $\rightarrow$ 1**

Ans. (C)

Sol. (1) $\alpha_1 = \binom{6}{3}\binom{5}{2} = 200$

So P $\rightarrow$ 4

(2) $\alpha_2 = \binom{6}{1}\binom{5}{1} + \binom{6}{2}\binom{5}{2} + \binom{6}{3}\binom{5}{3} + \binom{6}{4}\binom{5}{4} + \binom{6}{5}\binom{5}{5}$

$= \binom{11}{5} - 1$

= 46!

So Q → 6

(3) $\alpha_3 = \binom{5}{2}\binom{6}{3} + \binom{5}{3}\binom{6}{2} + \binom{5}{4}\binom{6}{1} + \binom{5}{5}\binom{6}{0}$

$= \binom{11}{5} - \binom{5}{0}\binom{6}{5} - \binom{5}{1}\binom{6}{4}$

= 381

So R → 5

(4) $\alpha_2 = \binom{5}{2}\binom{6}{2} - \binom{4}{1}\binom{5}{1} + \binom{5}{3}\binom{6}{1} - \binom{4}{2}\binom{1}{1} + \binom{5}{4} = 189$

So S → 2

17. Let $H : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a > b > 0$, be a hyperbola in the xy-plane whose conjugate axis LM subtends an angle of 60° at one of its vertices N. Let the area of the triangle LMN be $4\sqrt{3}$.

LIST-I	**LIST-II**
P. The length of the conjugate axis of H is	**1.** 8
Q. The eccentricity of H is	**2.** $\frac{4}{\sqrt{3}}$
R. The distance between the foci of H is	**3.** $\frac{2}{\sqrt{3}}$
S. The length of the latus rectum of H is	**4.** 4

The correct option is :

(A) **P → 4; Q → 2, R → 1; S → 3**

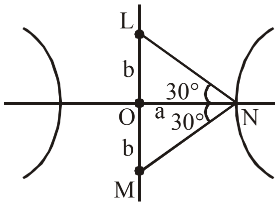
(B) **P → 4; Q → 3; R → 1; S → 2**

(C) **P → 4; Q → 1, R → 3; S → 2**

(D) **P → 3; Q → 4; R → 2; S → 1**

Ans. (B)

Sol.



$$\tan 30° = \frac{b}{a}$$

$\Rightarrow \quad a = b\sqrt{3}$

Now area of $\Delta LMN = \frac{1}{2}.2b.b\sqrt{3}$

$4\sqrt{3} = \sqrt{3}b^2$

$\Rightarrow \quad b = 2 \quad \& \quad a = 2\sqrt{3}$

$\Rightarrow \quad e = \sqrt{1+\frac{b^2}{a^2}} = \frac{2}{\sqrt{3}}$

P. Length of conjugate axis = 2b = 4

So $P \rightarrow 4$

Q. Eccentricity $e = \frac{2}{\sqrt{3}}$

So $Q \rightarrow 3$

R. Distance between foci = 2ae

$= 2(2\sqrt{3})\left(\frac{2}{\sqrt{3}}\right) = 8$

So $R \rightarrow 1$

S. Length of latus rectum $= \frac{2b^2}{a} = \frac{2(2)^2}{2\sqrt{3}} = \frac{4}{\sqrt{3}}$

So $S \rightarrow 2$

18. Let $f_1 : \mathbb{R} \to \mathbb{R}$, $f_2 : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, $f_3 : \left(-1, e^{\frac{\pi}{2}} - 2\right) \to \mathbb{R}$ and $f_4 : \mathbb{R} \to \mathbb{R}$ be functions defined

by

(i) $f_1(x) = \sin\left(\sqrt{1 - e^{-x^2}}\right)$

(ii) $f_2(x) = \begin{cases} \dfrac{|\sin x|}{\tan^{-1} x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$, where the inverse trigonometric function $\tan^{-1}x$ assumes values

in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

(iii) $f_3(x) = [\sin(\log_e(x + 2))]$, where for $t \in \mathbb{R}$, $[t]$ denotes the greatest integer less than or equal to t,

(iv) $f_4(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

List-I	**List-II**
P. the function f_1 is	**1.** **NOT** continuous at $x = 0$
Q. The function f_2 is	**2.** continuous at $x = 0$ and **NOT** differentiable at $x = 0$
R. The function f_3 is	**3.** differentiable at $x = 0$ and its derivative is **NOT** continuous at $x = 0$
S. The function f_4 is	**4.** differentiable at $x = 0$ and its derivative is continuous at $x = 0$

The correct option is :

(A) **P → 2; Q → 3, R → 1; S → 4**

(B) **P → 4; Q → 1; R → 2; S → 3**

(C) **P → 4; Q → 2, R → 1; S → 3**

(D) **P → 2; Q → 1; R → 4; S → 3**

Ans. (D)

Sol. (i) $f(x) = \sin\sqrt{1-e^{-x^2}}$

$$f_1'(x) = \cos\sqrt{1-e^{-x^2}} \cdot \frac{1}{2\sqrt{1-e^{-x^2}}}\left(0-e^{-x^2}.(-2x)\right)$$

at x = 0 $f_1'(x)$ does not exist

So. P → 2

(ii) $f_2(x) = \begin{cases} \dfrac{|\sin x|}{\tan^{-1} x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$

$$\lim_{x\to 0^+} \frac{\sin x}{x}\frac{x}{\tan^{-1} x} = 1$$

⇒ $f_2(x)$ does not continuous at x = 0

So Q → 1

(iii) $f_3(x) = \left[\sin \ell n(x+2)\right] = 0$

$1 < x + 2 < e^{\pi/2}$

⇒ $0 < \ell n(x + 2) < \frac{\pi}{2}$

⇒ $0 < \sin(\ell n(x + 2) < 1$

⇒ $f_3(x) = 0$

So R → 4

(iv) $f_4(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

So S → 3